

**ON TOPOLOGICAL AND LINEAR HOMEOMORPHISMS
OF CERTAIN FUNCTION SPACES**

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Let X be a countable metric space which is not locally compact. We prove that the function space $C_p(X)$ is homeomorphic to σ_ω . We also give examples of countable metric spaces X and Y which are not locally compact and such that $C_p(X)$ and $C_p(Y)$ are not linearly homeomorphic.

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Introduction

Let X be a space. Consider the spaces

$$C_p^*(X) = \{f \in \mathbb{R}^X \mid f \text{ is continuous and bounded}\}$$

and

$$C_p(X) = \{f \in \mathbb{R}^X \mid f \text{ is continuous}\}$$

as subspaces of \mathbb{R}^X .

In [6] van Mill showed that for a countable metric space which is not locally compact, $C_p^*(X) \approx \sigma_\omega$, where

$$\sigma_\omega = (l_f^2)^\infty \quad \text{and} \quad l_f^2 = \{x \in l^2 \mid x_i = 0 \text{ for all but finitely many } i\}$$

(l^2 denotes Hilbert space).

In his paper van Mill used results on Q -matrices and results of [8, 9]. The aim of this paper is to prove that $C_p(X) \approx \sigma_\omega$, by the same methods, and to give examples of countable metric spaces X and Y which are not locally compact such that $C_p(X)$ and $C_p(Y)$ are not linearly homeomorphic.

Observe that $C_p^*(X) \approx C_p(X)$ for X as above whereas these two spaces are not even uniformly isomorphic provided X is not compact.

Section 1 contains some definitions and theorems that we need in Section 2, where we will prove that $C_p(X) \approx \sigma_\omega$. In Section 2 we also sketch an alternative proof that $C_p(X) \approx \sigma_\omega$. In Section 3 we give examples of spaces X and Y such that $C_p(X)$ and $C_p(Y)$ are not linearly homeomorphic.

1. Preliminaries

Consider the Hilbert cube $Q = \prod_{i=1}^{\infty} [-1, 1]_i$ with the metric

$$d(x, y) = \sum_{i=1}^{\infty} 2^{-i} |x_i - y_i|.$$

A space which is homeomorphic to Q is called a *Hilbert cube*. If two spaces X and Y are homeomorphic we will use the symbol $X \approx Y$.

Let X and Y be compact spaces (by a space we mean a separable metric space). Put

$$C(X, Y) = \{f: X \rightarrow Y \mid f \text{ is continuous}\}$$

and

$$\mathcal{H}(Y) = \{f: Y \rightarrow Y \mid f \text{ is a homeomorphism}\}.$$

The topology on both spaces is derived from the metric

$$\hat{d}(f, g) = \sup\{d(f(x), g(x)) \mid x \in X\},$$

where d is an admissible metric on Y .

Let A be a closed subspace of X . A is a Z -set in X iff for every $f \in C(Q, X)$ and for every $\varepsilon > 0$, there is a $g \in C(Q, X)$ such that

- (1) $\hat{d}(f, g) < \varepsilon$,
- (2) $g(Q) \cap A = \emptyset$.

Notation: $A \in \mathcal{Z}(X)$.

1.1. Lemma. Let $A \subset Q$ with $\pi_j(A) \neq [-1, 1]$ for infinitely many j , then $A \in \mathcal{Z}(Q)$ ($\pi_j: Q \rightarrow [-1, 1]$ is the projection on the j th coordinate).

Let $\{A_n\}_{n \in \mathbb{N}}$ be an increasing family of Z -sets in X . Then $\{A_n\}_{n \in \mathbb{N}}$ is a *skeleton* in X iff for every $\varepsilon > 0$, for every $n \in \mathbb{N}$ and for every $Z \in \mathcal{Z}(X)$, there are $h \in \mathcal{H}(X)$ and $m \in \mathbb{N}$ such that

- (1) $\hat{d}(h, 1) < \varepsilon$,

- (2) $h|_{A_n} = 1$,
- (3) $h(Z) \subset A_m$.

The above definitions and the lemma can be found in [3]. The next three definitions are due to van Mill [6].

A \mathcal{L} -matrix in X is a collection $\mathcal{A} = \{A_m^n | n, m \in \mathbb{N}\}$ of Z -sets in X such that for every $m, n \in \mathbb{N}$,

- (1) $A_1^n = \emptyset$,
- (2) $A_m^n \subset A_{m+1}^n$,
- (3) $A_m^{n+1} \subset A_m^n$.

Define the *kernel* of \mathcal{A} by $\ker \mathcal{A} = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_m^n$.

Let $\mathcal{A} = \{A_m^n | n, m \in \mathbb{N}\}$ be a \mathcal{L} -matrix in Q . Then \mathcal{A} is a Q -matrix iff \mathcal{A} has the following properties:

- (1) $\forall n \in \mathbb{N}$: $\{A_m^n\}_{m \geq 1}$ is a skeleton in Q ,
- and $\forall n_1 < \dots < n_m \in \mathbb{N}$ and $\forall i_1, \dots, i_m \in \mathbb{N} \setminus \{1\}$:
- (2) $\bigcap_{k=1}^m A_{i_k}^{n_k} \approx Q$,
- (3) $\forall p \in \mathbb{N}$: $\{\bigcap_{k=1}^m A_{i_k}^{n_k} \cap A_{i_k}^{n_k+p}\}_{i \geq 1}$ is a skeleton in $\bigcap_{k=1}^m A_{i_k}^{n_k}$,
- (4) $\forall n \in \mathbb{N}$ and $\forall m \in \mathbb{N} \setminus \{1\}$: $\bigcap_{k=1}^m A_{i_k}^{n_k} \not\subset A_n^m \Rightarrow \bigcap_{k=1}^m A_{i_k}^{n_k} \cap A_n^m \in \mathcal{L}(\bigcap_{k=1}^m A_{i_k}^{n_k})$.

In [6] van Mill proved the following theorem.

1.2. Theorem. *If \mathcal{A} is a Q -matrix, then $\ker \mathcal{A} \approx \sigma_\omega$.*

Van Mill used this theorem to prove that if X is a countable metric space which is not locally compact, then $C_p^*(X) \approx \sigma_\omega$. The strategy of the proof is the following: First a nice subspace T of X is constructed and a Q -matrix \mathcal{B} is found such that $\ker \mathcal{B} \approx C_p^*(T)$. So by Theorem 1.2 it follows that $C_p^*(T) \approx \sigma_\omega$. Then by applying strong results of [8, 9] he uses this result to derive that $C_p^*(X) \approx \sigma_\omega$. By the same strategy we will prove that $C_p(X) \approx \sigma_\omega$.

Let $\mathcal{A} = \{A_m^n | n, m \in \mathbb{N}\}$ be a \mathcal{L} -matrix and let $A_{m_1}^{n_1}$ and $A_{m_2}^{n_2}$ be in \mathcal{A} such that $n_1 < n_2$ and $m_1 \geq m_2$. Then $A_{m_2}^{n_2} \subset A_{m_1}^{n_1}$ so $A_{m_1}^{n_1} \cap A_{m_2}^{n_2} = A_{m_2}^{n_2}$. So for $n_1 < \dots < n_m \in \mathbb{N}$ and $i_1, \dots, i_m \in \mathbb{N} \setminus \{1\}$ we may assume $i_1 < \dots < i_m$ if we are interested in $\bigcap_{k=1}^m A_{i_k}^{n_k}$.

The next theorem can be found in [3]. It will be used in Section 2.

1.3. Theorem. *If $\{A_i\}_{i \in \mathbb{N}}$ is an increasing family of Z -sets in Q such that*

- (1) $\forall i \in \mathbb{N}$: $A_i \in \mathcal{L}(A_{i+1})$,
- (2) $\forall i \in \mathbb{N}$: A_i is convex and infinite-dimensional,
- (3) $\bigcup_{i=1}^{\infty} A_i$ is dense in Q ,

then $\{A_i\}_{i \in \mathbb{N}}$ is a skeleton in Q .

2. Homeomorphic function spaces

In this section we will prove that for a countable metric space X which is not locally compact the function space $C_p(X)$ is homeomorphic to σ_ω . First we define

a test space T in the following way: $T = \mathbb{N}^2 \cup \{\infty\}$, where each point of \mathbb{N}^2 is isolated and $\{\{n, n+1, \dots\} \times \mathbb{N}\} \cup \{\infty\}$ is an open base at ∞ .

Let $C_{p,0}(T) = \{f \in C_p(T) \mid f(\infty) = 0\}$. We shall identify $C_p(T)$ with its subspace $\{f: T \rightarrow (-1, 1) \mid f \text{ is continuous}\}$. Let $P = \prod_{i=1}^{\infty} Q_i$, where $Q_i = Q$ for every $i \in \mathbb{N}$. Notice that P is a Hilbert cube. We can embed $C_{p,0}(T)$ into P by the embedding $\phi: C_{p,0}(T) \rightarrow P$ defined by

$$\phi(f)_i = f|_{\{i\} \times \mathbb{N}}.$$

Let $I = [-1, 1]$, $I_m = [-1 + 1/m, 1 - 1/m]$ for every $m \in \mathbb{N}$ and

$$B(\varepsilon) = \prod_{i=1}^{\infty} [-\varepsilon, \varepsilon]_i \quad \text{for every } \varepsilon > 0.$$

For every $n, m \in \mathbb{N}$ define $A_1^n = \emptyset$ and

$$A_m^n = \prod_{i=1}^m ((I_m)^n \times I \times I \times \dots)_i \times \prod_{i=m+1}^{\infty} B_i(2^{-n}) \subset \prod_{i=1}^{\infty} Q_i = P.$$

Let $\mathcal{A} = \{A_m^n \mid n, m \in \mathbb{N}\}$.

2.1. Lemma. $\ker \mathcal{A} = C_{p,0}(T)$.

Proof. Let $f \in \ker \mathcal{A}$ and $(i, j) \in \mathbb{N}^2$. By $f(i, j)$ we mean the j th coordinate of Q_i . Since $f \in \bigcup_{m=1}^{\infty} A_m^n$, there is $m \in \mathbb{N}$ with $f \in A_m^n$. If $i \leq m$, then $f(i, j) \in I_m \subset (-1, 1)$ and if $i > m$, then $f(i, j) \in [-2^{-j}, 2^{-j}] \subset (-1, 1)$. So f is well defined.

Now we prove that $f: T \rightarrow (-1, 1)$ is continuous. Therefore we only have to prove that f is continuous at ∞ . Let $\varepsilon > 0$ and $n \in \mathbb{N}$ such that $2^{-n} < \varepsilon$. Let $m \in \mathbb{N}$ such that $f \in A_m^n$. Then $|f(i, j)| \leq 2^{-n} < \varepsilon$ for $i > m$ and $j \in \mathbb{N}$. So f is continuous at ∞ . Conversely let $f \in C_{p,0}(T)$ and $n \in \mathbb{N}$. There is $m_1 \in \mathbb{N}$ with $|f(i, j)| < 2^{-n}$ for $i > m_1$ and $j \in \mathbb{N}$. There is $m_2 \in \mathbb{N}$ such that for every $i \leq m_1$ and $j \leq n$ we have $|f(i, j)| \leq 1 - 1/m_2$. Let $m = \max(m_1, m_2)$. Then $f \in A_m^n$. \square

2.2. Lemma. \mathcal{A} is a \mathcal{L} -matrix in P .

Proof. By Lemma 1.1 we have for every $n, m \in \mathbb{N}$ that $A_m^n \in \mathcal{L}(P)$. It is clear that for every $n, m \in \mathbb{N}$, $A_m^n \subset A_{m+1}^n$ and $A_m^{n+1} \subset A_m^n$. \square

2.3. Lemma. \mathcal{A} is a Q -matrix in P .

Proof. By Lemma 1.1 we have for every $\varepsilon > 0$ and $\delta < \varepsilon$ that $B(\delta) \in \mathcal{L}(B(\varepsilon))$.

Claim 1. $\forall n \in \mathbb{N}: \{A_m^n\}_{m>1}$ is a skeleton in P .

By Lemma 1.1 we have for every $n, m \in \mathbb{N}$ that $A_m^n \in \mathcal{L}(P)$ and $A_m^n \in \mathcal{L}(A_{m+1}^n)$. Because each A_m^n ($m > 2$) is a product of nondegenerate intervals, it is convex and infinite-dimensional. It is easy to verify that for every $n \in \mathbb{N}$, $\bigcup_{m=1}^{\infty} A_m^n$ is dense in P . By Theorem 1.3 we have for every $n \in \mathbb{N}$ that $\{A_m^n\}_{m>1}$ is a skeleton in P .

Now let $n_1 < \dots < n_m \in \mathbb{N}$ and $i_1, \dots, i_m \in \mathbb{N} \setminus \{1\}$. We may assume $i_1 < \dots < i_m$.

Claim 2. $\bigcap_{k=1}^m A_{i_k}^{n_k} \approx P$.

Because $A_{i_1}^{n_1} \subset \bigcap_{k=1}^m A_{i_k}^{n_k}$, $\bigcap_{k=1}^m A_{i_k}^{n_k}$ is a product of intervals and $A_{i_1}^{n_1} \approx P$ we have $\bigcap_{k=1}^m A_{i_k}^{n_k} \approx P$.

Claim 3. $\forall p \in \mathbb{N}: \{\bigcap_{k=1}^m A_{i_k}^{n_k} \cap A_{i_m+p}^{n_m+p}\}_{i \geq 1}$ is a skeleton in $\bigcap_{k=1}^m A_{i_k}^{n_k}$.

Let $p \in \mathbb{N}$ and $i \in \mathbb{N} \setminus \{1\}$. Let k be greater than $\max(i, i_m)$. The k th factor space of $\bigcap_{k=1}^m A_{i_k}^{n_k}$ is $B(2^{-n_m})$ and the k th factor space of $\bigcap_{k=1}^m A_{i_k}^{n_k} \cap A_{i_m+p}^{n_m+p}$ is $B(2^{-n_m-p})$, so by Lemma 2.1 we have $\bigcap_{k=1}^m A_{i_k}^{n_k} \cap A_{i_m+p}^{n_m+p} \in \mathcal{L}(\bigcap_{k=1}^m A_{i_k}^{n_k})$. If $i \geq i_m$, then the $(i+1)$ th factor space of $\bigcap_{k=1}^m A_{i_k}^{n_k} \cap A_{i_m+p}^{n_m+p}$ is $B(2^{-n_m-p})$ and the $(i+1)$ th factor space of $\bigcap_{k=1}^m A_{i_k}^{n_k} \cap A_{i_m+p}^{n_m+p}$ is $B(2^{-n_m})$. If there is an $l \leq m$ such that $i_{l-1} < i+1 \leq i_l$ ($i_0 = 1$), then the $(i+1)$ th factor space of $\bigcap_{k=1}^m A_{i_k}^{n_k} \cap A_{i_m+p}^{n_m+p}$ is a Z -set in the $(i+1)$ th factor space of $\bigcap_{k=1}^m A_{i_k}^{n_k} \cap A_{i_m+p}^{n_m+p}$. We conclude that for every $i \in \mathbb{N} \setminus \{1\}$, $\bigcap_{k=1}^m A_{i_k}^{n_k} \cap A_{i_m+p}^{n_m+p} \in \mathcal{L}(\bigcap_{k=1}^m A_{i_k}^{n_k} \cap A_{i_m+p}^{n_m+p})$. The rest of the claim can be proved as in Claim 1.

Claim 4. $\forall s \in \mathbb{N}$ and $\forall t \in \mathbb{N} \setminus \{1\}$:

$$\bigcap_{k=1}^m A_{i_k}^{n_k} \not\subset A_t^s \Rightarrow \bigcap_{k=1}^m A_{i_k}^{n_k} \cap A_t^s \in \mathcal{L}\left(\bigcap_{k=1}^m A_{i_k}^{n_k}\right).$$

If $s > n_m$, then $\bigcap_{k=1}^m A_{i_k}^{n_k} \cap A_t^s \in \mathcal{L}(\bigcap_{k=1}^m A_{i_k}^{n_k})$ by Claim 3. If $s \leq n_m$, there is $k \leq m$ such that $n_{k-1} < s \leq n_k$ (let $n_0 = 0$). This implies $t < i_k$. So there is $l \in \mathbb{N}$ such that $i_{l-1} < t+1 \leq i_l$ ($i_0 = 0$). The $(t+1)$ th factor space of $\bigcap_{k=1}^m A_{i_k}^{n_k}$ is $B(2^{-n_{l-1}})$ and the $(t+1)$ th factor space of $\bigcap_{k=1}^m A_{i_k}^{n_k} \cap A_t^s$ is $B(2^{-s})$, because $s > n_{l-1}$. So $\bigcap_{k=1}^m A_{i_k}^{n_k} \cap A_t^s \in \mathcal{L}(\bigcap_{k=1}^m A_{i_k}^{n_k})$.

By Claims 1-4 we have that \mathcal{A} is a Q -matrix. \square

2.4. Corollary. $C_{p,0}(T) \approx \sigma_\omega$.

Proof. This follows immediately from the Lemmas 2.1, 2.3 and Theorem 1.2. \square

2.5. Theorem. Let X be a countable metric space which is not locally compact. Then $C_p(X) \approx \sigma_\omega$.

Proof. In [6] van Mill proved that $C_{p,0}^*(T) \approx \sigma_\omega$, where

$$C_{p,0}^*(T) = \{f \in C_p^*(T) \mid f(\infty) = 0\}.$$

By using results of [8, 9], he derived from this fact that $C_p^*(X) \approx \sigma_\omega$. By using the same technique it follows from Corollary 2.4 that $C_p(X) \approx \sigma_\omega$. \square

Remark. Let X be a countable metric space which is not locally compact at x_0 . It is possible to find a Q -matrix \mathcal{A} such that $\ker \mathcal{A} = C_{p,0}(X)$, where

$$C_{p,0}(X) = \{f \in C_p(X) \mid f(x_0) = 0\}.$$

From this it follows that $C_p(X) \approx C_{p,0}(X) \times \mathbb{R} \approx \sigma_\omega \times \mathbb{R} \approx \sigma_\omega$. The same can be done for $C_p^*(X)$. These results can be found in [2].

Remark. We are indebted to the referee for providing us with the Q -matrix for $C_{p,0}(T)$ presented in this section. This Q -matrix is much simpler than the one originally constructed in [2].

3. Function spaces that are not linearly homeomorphic

In Section 2 we proved that for countable metric spaces X and Y which are both not locally compact, the spaces $C_p(X)$ and $C_p(Y)$ are homeomorphic. In the present section we shall construct countable metric spaces X_1 , X_2 and X_3 which are not locally compact, such that $C_p(X_i)$ and $C_p(X_j)$ are not *linearly* homeomorphic for $i \neq j$. First we derive the following theorem.

3.1. Theorem. *Let M be a countable metric space which is not locally compact. If M contains an infinite closed discrete set of non isolated points, then $C_p(T)$ and $C_p(M)$ are not linearly homeomorphic.*

Proof. Let $A = \{x_1, x_2, \dots\}$ be a closed discrete set of non isolated points in M . For every $n \in \mathbb{N}$, let $\{U_j^n \mid j \in \mathbb{N}\}$ be a clopen base for x_n , such that $U_{j_1}^{n_1} \cap U_{j_2}^{n_2} = \emptyset$ if $n_1 \neq n_2$ and $j_1, j_2 \in \mathbb{N}$ (this is possible since A is closed discrete and M is zero-dimensional).

Now suppose that $\phi : C_p(M) \rightarrow C_{p,0}(T)$ is a linear homeomorphism. Let g_j^n be the characteristic function of U_j^n on M . Since U_j^n is clopen, $g_j^n \in C_p(M)$. Furthermore let $h_j^n = \phi(g_j^n) \in C_{p,0}(T)$.

Claim 1. For every $n \in \mathbb{N}$ and for every $t \in T$, the set $\{h_j^n(t) \mid j \in \mathbb{N}\}$ is bounded.

Suppose the contrary. Without loss of generality we may assume that $h_j^n(t) \geq 0$ for every $j \in \mathbb{N}$. Then for every $k \in \mathbb{N}$, there is $j_k \in \mathbb{N}$ such that $h_{j_k}^n(t) \geq 2^k$. Notice that $f = \sum_{k=1}^{\infty} 2^{-k} g_{j_k}^n \in C_p(M)$. But then $\phi(f) = \sum_{k=1}^{\infty} 2^{-k} h_{j_k}^n \in C_{p,0}(T)$. Since $\phi(f)(t) = \sum_{k=1}^{\infty} 2^{-k} h_{j_k}^n(t)$ is divergent, we have a contradiction.

Claim 2. For every $t \in T$ there are only finitely many $n \in \mathbb{N}$ such that there is $j_n \in \mathbb{N}$ with $h_{j_n}^n(t) \neq 0$.

Suppose there is $t \in T$ such that there are infinitely many $n \in \mathbb{N}$, say n_1, n_2, \dots with the property that for every $i \in \mathbb{N}$, there is $j_i \in \mathbb{N}$ such that $h_{j_i}^{n_i}(t) \neq 0$. Without loss of generality we may assume that $h_{j_i}^{n_i}(t) > 0$ for every $i \in \mathbb{N}$. Let $\lambda_i = (h_{j_i}^{n_i}(t))^{-1}$. Notice that $f = \sum_{i=1}^{\infty} \lambda_i g_{j_i}^{n_i} \in C_p(M)$, so $\phi(f) = \sum_{i=1}^{\infty} \lambda_i h_{j_i}^{n_i} \in C_{p,0}(T)$. Since $\phi(f)(t) = \sum_{i=1}^{\infty} \lambda_i h_{j_i}^{n_i}(t) = \sum_{i=1}^{\infty} 1$ is divergent, we have a contradiction.

Now let $n \in \mathbb{N}$. Then $g_j^n \rightarrow \chi_{\{x_n\}}$ (the characteristic function of $\{x_n\}$) pointwise ($j \rightarrow \infty$). Observe that $\chi_{\{x_n\}}$ is not continuous, since x_n is a non isolated point. For every $n \in \mathbb{N}$ we define a sequence $(f_k^n)_{k \in \mathbb{N}}$ in $C_{p,0}(T)$ as follows: Since T is countable, we can enumerate the elements of $T \setminus \{\infty\}$ as $\{t_1, t_2, \dots\}$. Inductively for every $l \in \mathbb{N}$, we find converging sequences $(h_{j_k}^n(t_l))_{k \in \mathbb{N}}$ as follows: Since $\{h_j^n(t_l) \mid j \in \mathbb{N}\}$ is bounded (Claim 1), there is a converging subsequence $(h_{j_k}^n(t_l))_{k \in \mathbb{N}}$. Suppose the sequence is found for $i = 1, \dots, l$. Since $\{h_{j_k}^n(t_{l+1}) \mid k \in \mathbb{N}\}$ is bounded (Claim 1), there is a converging subsequence $(h_{j_k^{l+1}}^n(t_{l+1}))_{k \in \mathbb{N}}$.

Now let $f_k^n = h_{j_k}^n$ ($k \in \mathbb{N}$). By construction, for every $t \in T$, $\sigma_n(t) = \lim_{k \rightarrow \infty} f_k^n(t)$ exists. So $\sigma_n: T \rightarrow \mathbb{R}$ is well defined. Observe that $\sigma_n(\infty) = 0$. Suppose σ_n is continuous. Then $f_k^n \rightarrow \sigma_n$ in $C_{p,0}(T)$. Since ϕ^{-1} is continuous, $\phi^{-1}(f_k^n) = g_{j_k}^n \rightarrow \phi^{-1}(\sigma_n)$ in $C_p(M)$, so $\chi_{\{x_n\}} = \lim_{k \rightarrow \infty} g_{j_k}^n$ is continuous; a contradiction.

Since σ_n is well defined and $T \setminus \{\infty\}$ is discrete, σ_n is discontinuous at ∞ . It follows that there is a sequence $(y_l^n)_{l \in \mathbb{N}}$ in T , converging to ∞ such that $|\sigma_n(y_l^n)| > \varepsilon_n$ for some $\varepsilon_n > 0$ and for every $l \in \mathbb{N}$. Since ϕ is linear we may assume that $\varepsilon_n = 1$ for every $n \in \mathbb{N}$.

We now inductively construct sequences $(n_i)_{i \in \mathbb{N}}$, $(k_i)_{i \in \mathbb{N}}$ in \mathbb{N} and $(t_i)_{i \in \mathbb{N}}$ in T such that

- (i) $n_1 < n_2 < \dots$,
- (ii) $t_i \rightarrow \infty$,
- (iii) $|f_{k_i}^{n_i}(t_i)| > 1$ for every $i \in \mathbb{N}$,
- (iv) $f_{k_i}^{n_i}(t_j) = 0$ for every $i \in \mathbb{N}$ and $j < i$,
- (v) $|f_{k_i}^{n_i}(t_i)| < 1/(2(i-1))$ for every $i \in \mathbb{N}$ and $j < i$,

as follows:

Let $n_1 = 1$. Since $\lim_{k \rightarrow \infty} |f_k^{n_1}(y_1^{n_1})| = |\sigma_{n_1}(y_1^{n_1})| > 1$, there is $k_1 \in \mathbb{N}$ such that $|f_{k_1}^{n_1}(y_1^{n_1})| > 1$. Let $t_1 = y_1^{n_1}$.

Suppose $n_1, \dots, n_i, k_1, \dots, k_i$ and t_1, \dots, t_i are found. By Claim 2, for every $j \leq i$ there are only finitely many $n \in \mathbb{N}$ such that $f_m^n(t_j) \neq 0$ for some $m \in \mathbb{N}$. It follows that there is $n_{i+1} > n_i$ such that for every $j \leq i$ and $m \in \mathbb{N}$, $f_m^{n_{i+1}}(t_j) = 0$, so (i) and (iv) are satisfied. Since $y_l^{n_{i+1}} \rightarrow \infty$ ($l \rightarrow \infty$) and $f_{k_l}^{n_{i+1}} \in C_{p,0}(T)$ for $j \leq i$, there is $l_0 \in \mathbb{N}$ such that for $t_{i+1} = y_{l_0}^{n_{i+1}}$ we have

$$|f_{k_j}^{n_i}(t_{i+1})| < 1/(2i) \quad (j \leq i),$$

and the first coordinate of t_{i+1} is greater than the first coordinates of t_1, \dots, t_i . With this t_{i+1} (v) is satisfied.

Finally, since $\lim_{k \rightarrow \infty} |f_k^{n_{i+1}}(t_{i+1})| = |\sigma_{n_{i+1}}(t_{i+1})| > 1$, there is $k_{i+1} \in \mathbb{N}$ such that

$$|f_{k_{i+1}}^{n_{i+1}}(t_{i+1})| > 1,$$

so (iii) is satisfied and the induction is completed. By construction (ii) is also satisfied.

Notice that by (i), $f = \sum_{j=1}^{\infty} \phi^{-1}(f_{k_j}^{n_j}) \in C_p(M)$. So $\phi(f) = \sum_{j=1}^{\infty} f_{k_j}^{n_j} \in C_{p,0}(T)$. Since by (ii) $t_i \rightarrow \infty$, $\phi(f)(t_i) \rightarrow 0$. But

$$\begin{aligned} |\phi(f)(t_i)| &= \left| \sum_{j=1}^{\infty} f_{k_j}^{n_j}(t_i) \right| \\ &= \left| \sum_{j=1}^{i-1} f_{k_j}^{n_j}(t_i) + f_{k_i}^{n_i}(t_i) \right| \quad (\text{by (iv)}) \\ &\geq \left| f_{k_i}^{n_i}(t_i) \right| - \left| \sum_{j=1}^{i-1} f_{k_j}^{n_j}(t_i) \right| \\ &> 1 - (i-1)/(2(i-1)) \quad (\text{by (iii) and (v)}) \\ &= 1 - \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

A contradiction.

Since $C_{p,0}(T)$ and $C_p(T)$ are linearly homeomorphic, we proved the theorem. \square

Now let \mathbb{Q} be the set of rationals and $\sum T = \sum_{i=1}^{\infty} T_i$ the topological sum of infinitely many copies of T . For convenience let ∞_i be the non isolated point in T_i . Notice that \mathbb{Q} and $\sum T$ are countable metric spaces which are not locally compact. By Theorem 3.1 we have the following corollary.

3.2. Corollary. (a) $C_p(\mathbb{Q})$ and $C_p(T)$ are not linearly homeomorphic.
 (b) $C_p(\sum T)$ and $C_p(T)$ are not linearly homeomorphic.

However Theorem 3.1 does not decide whether $C_p(\mathbb{Q})$ and $C_p(\sum T)$ are linearly homeomorphic. In the sequel we will show that this is not the case. First we need some results from [1].

For every space X let $C(X) = \{f: X \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$. If we endow $C(X)$ with the compact-open topology we write $C_0(X)$. A subbase for $C_0(X)$ is

$$\{A_X(K, U) \mid K \subset X \text{ compact and } U \subset \mathbb{R} \text{ open}\},$$

where $A_X(K, U) = \{f \in C(X) \mid f(K) \subset U\}$.

3.3. Theorem (Arhangel'skiĭ [1]). *If $\theta: C_p(X) \rightarrow C_p(Y)$ is a linear homeomorphism, then θ considered as a function from $C_0(X) \rightarrow C_0(Y)$ is also a linear homeomorphism.*

Let X and Y be spaces and $\theta: C(X) \rightarrow C(Y)$ a linear mapping.

3.4. Definition. For every $y \in Y$, the *support* of y in X is defined to be the set $\text{supp}(y)$ of all $x \in X$ satisfying the condition that for every neighborhood U of $x \in X$ there is an $f \in C(X)$ such that $f(X \setminus U) \subset \{0\}$ and $\theta(f)(y) \neq 0$.

Notice that for every $y \in Y$, $\text{supp}(y)$ is closed.

3.5. Definition. θ is said to be *effective* if for every $f, g \in C(X)$ and $y \in Y$, such that f and g coincide on a neighborhood of $\text{supp}(y)$, $\theta(f)(y) = \theta(g)(y)$.

3.6. Proposition (Arhangel'skiĭ [1]). *Let $\theta: C_0(X) \rightarrow C_0(Y)$ be a linear homeomorphism. Then*

- (a) θ is effective,
- (b) for every compact $K \subset Y$ we have that $\overline{\bigcup_{y \in K} \text{supp}(y)}$ is compact.

Remark. In fact Arhangel'skiĭ proved that for every bounded set $K \subset Y$ (that means for every continuous real-valued f on Y , $f(K)$ is bounded in \mathbb{R}), $\bigcup_{y \in K} \text{supp}(y)$ is bounded in X . For metric spaces we then have the formulation in Proposition 3.6(b).

We shall prove that $C_p(\sum T)$ and $C_p(\mathbb{Q})$ are not linearly homeomorphic. To derive a contradiction we assume a linear homeomorphism $\theta: C_0(\sum T) \rightarrow C_0(\mathbb{Q})$. For the proof we need a property which can be found in [4].

3.7. Definition. Let E be a Banach space and $(x_n)_{n \in \mathbb{N}}$ a sequence in E . We say that $x_n \rightarrow x$ weakly iff $f(x_n) \rightarrow f(x)$ for every (norm) continuous linear functional f on E .

3.8. Definition. A Banach space E has the *weak Banach–Saks property* iff for every sequence $(x_n)_{n \in \mathbb{N}}$ in E such that $x_n \rightarrow 0$ weakly, there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ with $\|\sum_{k=1}^j x_{n_k}/j\| \rightarrow 0$.

It is easy to see that the weak Banach–Saks property is a property which is preserved by linear homeomorphisms and which is hereditary.

For a space X we denote $X^{(1)}$ the set of accumulation points of X . Inductively we can define $X^{(n)} = (X^{(n-1)})^{(1)}$ for every $n \in \mathbb{N}$ and $X^{(\omega)} = \bigcap_{n=1}^{\infty} X^{(n)}$.

3.9. Theorem [4, p. 85]. *Let X be a compact space. Then $C_0(X)$ has the weak Banach–Saks property iff $X^{(\omega)} = \emptyset$.*

Let K be a copy of $\omega^\omega + 1$ in \mathbb{Q} and $L = \overline{\bigcup_{y \in K} \text{supp}(y)}$. Notice that K is compact and therefore by Proposition 3.6(b), L is compact. This means that there is $p \in \mathbb{N}$ such that $L \subset \sum_{i=1}^p T_i$. Furthermore by Theorem 3.9, $C_0(L)$ has and $C_0(K)$ does not have the weak Banach–Saks property.

3.10. Lemma. *For every $f, g \in C_0(\mathbb{Q})$ with $f|K \neq g|K$ it follows that*

$$\theta^{-1}(f)|L \neq \theta^{-1}(g)|L.$$

Proof. Let $y \in K$ be such that $f(y) \neq g(y)$ and let R_0 and R_1 be disjoint open neighborhoods of $f(y)$ and $g(y)$, respectively. Then $A_{\mathbb{Q}}(\{y\}, R_0)$ and $A_{\mathbb{Q}}(\{y\}, R_1)$ are disjoint open neighborhoods of f and g in $C_0(\mathbb{Q})$. So $\theta^{-1}(A_{\mathbb{Q}}(\{y\}, R_0))$ and $\theta^{-1}(A_{\mathbb{Q}}(\{y\}, R_1))$ are disjoint open neighborhoods of $\theta^{-1}(f)$ and $\theta^{-1}(g)$ in $C_0(\sum T)$. There consequently exist compact $K_1, \dots, K_n, L_1, \dots, L_m \subset \sum T$ and open $U_1, \dots, U_n, V_1, \dots, V_m \subset \mathbb{R}$ such that

$$\theta^{-1}(f) \in \bigcap_{i=1}^n A_{\sum T}(K_i, U_i) \subset \theta^{-1}(A_{\mathbb{Q}}(\{y\}, R_0)),$$

and

$$\theta^{-1}(g) \in \bigcap_{j=1}^m A_{\sum T}(L_j, V_j) \subset \theta^{-1}(A_{\mathbb{Q}}(\{y\}, R_1)).$$

We claim there is a $z \in \text{supp}(y) \subset L$ such that $\theta^{-1}(f)(z) \neq \theta^{-1}(g)(z)$ (and then we are done). Striving for a contradiction, assume the contrary.

For every $s \leq p$ let $I_s = \{i \leq n \mid \infty_s \notin K_i\}$, $J_s = \{j \leq m \mid \infty_s \notin L_j\}$ and

$$P_s = \bigcup_{i \in I_s} (K_i \cap T_s) \cup \bigcup_{j \in J_s} (L_j \cap T_s).$$

Then P_s is compact in T_s and $\infty_s \notin P_s$, so P_s is finite. Let $P = \bigcup_{s \leq p} P_s$. Then P is finite and $P \cap \{\infty_1, \dots, \infty_p\} = \emptyset$.

Define $f' : \sum T \rightarrow \mathbb{R}$ by

$$f'(x) = \begin{cases} \theta^{-1}(f)(x) & \text{if } x \in P \cup \sum_{i=p+1}^{\infty} T_i, \\ \theta^{-1}(f)(\infty_s) & \text{if } x \in T_s \setminus P_s \text{ for } s \leq p. \end{cases}$$

Define $g' : \sum T \rightarrow \mathbb{R}$ by

$$g'(x) = \begin{cases} \theta^{-1}(g)(x) & \text{if } x \in P \cup \sum_{i=p+1}^{\infty} T_i, \\ \theta^{-1}(g)(\infty_s) & \text{if } x \in T_s \setminus P_s \text{ for } s \leq p. \end{cases}$$

Then f' and g' are continuous because $T_s \setminus P_s$ is a neighborhood of ∞_s .

Claim 1. f' and g' coincide on a neighborhood of $\text{supp}(y)$.

Let $M = \{s \leq p \mid \infty_s \in \text{supp}(y)\}$. Then by assumption we have that for every $s \in M$ $\theta^{-1}(f)(\infty_s) = \theta^{-1}(g)(\infty_s)$. Let $U = \sum_{s \in M} T_s \setminus P_s \cup \text{supp}(y)$. Then U is a neighborhood of $\text{supp}(y)$ on which f' and g' coincide.

Now by Claim 1 and the fact that θ is effective we have $\theta(f')(y) = \theta(g')(y)$.

Claim 2. $f' \in \bigcap_{i=1}^n A_{\sum T}(K_i, U_i)$ and $g' \in \bigcap_{j=1}^m A_{\sum T}(L_j, V_j)$.

Let $i \leq n$ and $x \in K_i$. If $x \in P \cup \sum_{i=p+1}^{\infty} T_i$, then $f'(x) = \theta^{-1}(f)(x) \in U_i$ because $\theta^{-1}(f) \in \bigcap_{i=1}^n A_{\sum T}(K_i, U_i)$, and if $x \notin P \cup \sum_{i=p+1}^{\infty} T_i$ we have $x \in T_s \setminus P_s$ for some $s \leq p$. Because $x \in K_i \cap T_s$ and $x \notin P_s$ we have $\infty_s \in K_i$, so $f'(x) = \theta^{-1}(f)(\infty_s) \in U_i$. The remaining part of the claim can be proved similarly.

Now we have $\theta(f') \in A_{\mathbb{Q}}(\{y\}, R_0)$ and $\theta(g') \in A_{\mathbb{Q}}(\{y\}, R_1)$. But this means $\theta(f')(y) \neq \theta(g')(y)$; a contradiction. \square

Because $K \subset \mathbb{Q}$ is compact and $L \subset \sum T$ is compact we can find retractions $r : \mathbb{Q} \rightarrow K$ and $s : \sum T \rightarrow L$ (see [5]). Define

$$\psi : C_0(K) \rightarrow C_0(L) \quad \text{by } \psi(f) = \theta^{-1}(f \circ r)|L,$$

and

$$\phi : C_0(L) \rightarrow C_0(K) \quad \text{by } \phi(g) = \theta(g \circ s)|K.$$

3.11. Lemma. ψ is a linear embedding.

Proof. It is easy to see that ψ and ϕ are well-defined continuous functions. We claim that for every $h \in C_0(K)$ we have $\phi(\psi(h)) = h$. To the contrary suppose $\theta(\psi(h) \circ s)|K \neq h = (h \circ r)|K$. By Lemma 3.10 we have $(\psi(h) \circ s)|L \neq \theta^{-1}(h \circ r)|L$. But this implies $\psi(h) \neq \psi(h)$; a contradiction. It easily follows that ψ is a linear embedding. \square

3.12. Proposition. $C_p(\sum T)$ and $C_p(\mathbb{Q})$ are not linearly homeomorphic.

Proof. Suppose $C_p(\sum T)$ and $C_p(\mathbb{Q})$ are linearly homeomorphic. By Theorem 3.3 $C_0(\sum T)$ and $C_0(\mathbb{Q})$ are linearly homeomorphic. Then by Lemma 3.11 we have a linear embedding $\psi: C_0(K) \rightarrow C_0(L)$. $C_0(L)$ has the weak Banach-Saks property which $C_0(K)$ does not have; a contradiction. \square

Remark. In [7] Pelant proved that the function spaces $C_p^*(T)$ and $C_p^*(\mathbb{Q})$ are not linearly homeomorphic. His proof does not seem to generalize to get our result that $C_p(T)$ and $C_p(\mathbb{Q})$ are not linearly homeomorphic.

Note added in proof

Dobrowolski, Gulko and Mogilski recently proved that $C_p(X) \approx \sigma_\omega$ provided X is a countable nondiscrete metric space. This result generalizes Theorem 2.5 of the present paper.

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